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COLUMN INSTABILITY UNDER NONCONSERVATIVE FORCES, WITH INTERNAL AND EXTERNAL DAMPING -- FINITE ELEMENT USING ADJOINT VARIATIONAL PRINCIPLES

Julian J. Wu

Watervliet Arsenal
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13. ABSTRACT

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Cross-Reference Data

Columns

Stability

Vibrations

Damping

Finite Element

Nonadjoint Variational Principles

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LIST OF CAPTIONS

1. INTRODUCTION

The problems of elastic stability under nonconservative forces have attracted many researchers in the last two decades. A review paper by G. Herrmann [1] contains a comprehensive bibliography of work performed before 1967. The practical significance of this class of problems has become increasingly noted with the advancement of modern technology. A flexible missile under thrust and a structural part of an aircraft under aerodynamic forces are obvious examples. Approximate methods are usually required for the solutions of such problems. Galerkin's method was first used by H. Leipholz [2] to obtain solutions for a nonconservative system. Because boundary value problems in the theory of nonconservative elastic stability are nonself-adjoint, by virtue of the fact that nonconservative forces do not possess potentials, no complete functional exists for the classical form of Hamilton's principle. M. Levinson [3] points out later that the conventional Hamilton's principle with nonconservative forces can be written as a well-posed variational principle with some constraint conditions and the Ritz method can thus be employed, formally at least, for the solution of these problems. A more recent trend has been the interest in the establishment of well-posed variational principles without any constraint conditions by the introduction of the adjoint systems [4-10].

Based on the variational principle suggested by Levinson, the finite element method has been employed for solutions of several nonconservative stability problems without damping [11,12,13]. In contrast, the present work is an application of the variational principle of Leipholz' type [5]. The effects due to both the internal and external damping are also included

in this study. The finite element method has been shown to be of the Ritz type of approximations in the linear theory of elasticity [14,15,16]. A variational principle is therefore a natural basis for its formulations. The effect of damping on various nonconservative stability problems has been studied by many investigators [9,10,17-34]. The difficulty arises not only from the nonself-adjoint nature of the differential equations but also from the fact that the eigenvalues involved are complex numbers in general.

Two classical problems with both internal and external damping terms are considered in this work, i.e., the Leipholz' problem [2] (a cantilevered column with uniformly distributed force tangential along the column's length) and the Beck's problem [35] (a cantilevered column subject to a follower force at its free end). The internal damping is assumed to result from a Kelvin-Voigt type of material and the external damping, due to a dissipative force proportional to the velocity at a point in the column. In the next section, the differential equations and boundary conditions of the problems and their adjoints are defined. The derivation of the respective adjoint variational principles is also outlined here. In Section 3, we give the finite element formulations that lead to the matrix eigen-equations. Final results and discussion on the numerical methods employed in this work are presented in Sections 4 and 5.

2. STATEMENT OF THE PROBLEM AND THE BASIS OF SOLUTIONS

The configurations and loads of Leipholz' problem and of Beck's are shown in Figures 1(a) and 1(b). In the absence of damping, the differential equation for Leipholz problem [2] is

$$EI \frac{\partial^4 \bar{u}}{\partial \bar{x}^4} + q(\ell - \bar{x}) \frac{\partial^2 \bar{u}}{\partial \bar{x}^2} + \rho A \frac{\partial^2 \bar{u}}{\partial \bar{t}^2} = 0,$$

and for Beck's problem [33],

$$EI \frac{\partial^4 \bar{u}}{\partial \bar{x}^4} + P \frac{\partial^2 \bar{u}}{\partial \bar{x}^2} + \rho A \frac{\partial^2 \bar{u}}{\partial \bar{t}^2} = 0, \tag{2}$$

with boundary conditions for both cases:

$$\left.\begin{aligned} \bar{u}\Big|_{\bar{x}=0} &= \frac{\partial \bar{u}}{\partial \bar{x}}\Big|_{\bar{x}=0} = 0 \\ \text{and} \quad \frac{\partial^2 \bar{u}}{\partial \bar{x}^2}\Big|_{\bar{x}=\ell} &= \frac{\partial^3 \bar{u}}{\partial \bar{x}^3}\Big|_{\bar{x}=\ell} = 0 \end{aligned}\right\} \tag{3}$$

where $\bar{u} = \bar{u}(\bar{x},\bar{t})$ is the deflection of the column from its undisturbed postion; $\bar{x},\bar{t}$ are the spatial and time variables, respectively; E, the Young's modulus, ρ, the density; I, the second moment and A, the area of the cross-section. The letter q denotes the force per unit length in the Leipholz' problem and P the concentrated follower force in Beck's problem.

To include damping effects, a dissipative for F_d, such as the air resistance, is assumed to be proportional to the velocity of the column at a point, i.e.,

$$F_d = - \beta_1 \frac{\partial \bar{u}}{\partial \bar{t}} \tag{4}$$

where β_1 is the external damping constant. The material is assumed to be of a linear viscoelastic solid of a Kelvin-Voigt type. The one dimensional stress-strain relation can be written as

$$\sigma = E\,\varepsilon + E^* \frac{d\varepsilon}{d\bar{t}} \tag{5}$$

where σ, ε are the one-dimensional stress and strain respectively; and E^*, the viscosity of the material.

The equations of motion, including damping terms, are then [9]

$$EI \frac{\partial^4 \bar{u}}{\partial \bar{x}^4} + E^*I \frac{\partial^5 \bar{u}}{\partial \bar{x}^4 d\bar{t}} + q(\ell - \bar{x}) \frac{\partial^2 \bar{u}}{\partial \bar{x}^2} + \beta_1 \frac{\partial \bar{u}}{\partial \bar{t}} + \rho A \frac{\partial^2 \bar{u}}{\partial \bar{t}^2} = 0 \tag{6}$$

for Leipholz' problem and

$$EI \frac{\partial^4 \bar{u}}{\partial \bar{x}^4} + E^*I \frac{\partial^5 \bar{u}}{\partial \bar{x}^4 \partial \bar{t}} + P \frac{\partial^2 \bar{u}}{\partial \bar{x}^2} + \beta_1 \frac{\partial \bar{u}}{\partial \bar{t}} + \rho A \frac{\partial^2 \bar{u}}{\partial \bar{t}^2} = 0 \tag{7}$$

for Beck's problem.

It will be convenient to use nondimensional quantities for the analysis. Thus, let

$$\left.\begin{aligned} x &= \frac{\bar{x}}{\ell}, \qquad & t &= \frac{\bar{t}}{c}, \qquad & u &= \frac{\bar{u}}{\ell} \\ \eta &= \frac{E^*}{Ec}, \qquad & \beta &= \frac{\beta_1 \ell^4}{EIc}, \qquad & c &= \frac{\rho A \ell^4}{EI} \end{aligned}\right\} \tag{8}$$

where the constant c has the dimension of time and all other quantities in Eqs. (8) are dimensionless. The nondimensionalized load parameters are

$$Q = \frac{q\ell^3}{EI}$$

and

$$Q = \frac{P\ell^2}{EI} \tag{9}$$

for Leipholz' and Beck's problem respectively. Using the notation defined in Eqs. (8) and (9), the nondimensionalized differential equations become

$$u^{IV} + \eta \dot{u}^{IV} + Q(1-x)u'' + \beta\dot{u} + \ddot{u} = 0 \tag{10}$$

for Leipholz problem and

$$u^{IV} + \eta\, \dot{u}^{IV} + Q\, \ddot{u} + \beta\dot{u} + \ddot{u} = 0 \tag{11}$$

for Beck's problem. The boundary conditions are now,

$$\left.\begin{aligned} u\big|_{x=0} &= u'\big|_{x=0} = 0 \\ [u'' + \eta\dot{u}'']_{x=1} &= [u''' + \eta\,\dot{u}''']_{x=1} = 0 \end{aligned}\right\} \tag{12}$$

for both cases. Here we have adapted the usual notation that a prime (') or Roman numeral denotes a differentiation with respect to x and a dot (·) denotes differentiation with respect to t. The problems adjoint to these two have been shown [9] to be

$$v^{IV} - \eta\, \dot{v}^{IV} + [Q(1-x)v]'' + \ddot{v} - \beta\, \dot{v} = 0 \tag{13}$$

$$\left.\begin{aligned} v\big|_{x=0} &= v'\big|_{x=0} = 0 \\ [v'' - \eta\, \dot{v}'']_{x=1} &= [v''' - \eta\, \dot{v}''']_{x=1} = 0 \end{aligned}\right\} \tag{14}$$

for the adjoint to the Leipholz problem, and

$$v^{IV} - \eta \dot{v}^{IV} + Qv'' + \ddot{v} - \beta \dot{v} = 0 \tag{15}$$

$$\left.\begin{aligned} v\big|_{x=0} &= v'\big|_{x=0} = 0 \\ [v'' - \eta \dot{v}'' + Qv]_{x=1} &= [v''' - \eta \dot{v}''' + Qv']_{x=1} = 0 \end{aligned}\right\} \tag{16}$$

for the adjoint to Beck's problem, where v denotes the adjoint field variable.

For Leipholz' problem and its adjoint, one can consider Eqs. (10) and (13) as the dynamic equations of equilibrium of the original and the adjoint system. Introducing virtual displacements δu and δv which satisfy the respective end conditions, it is then possible to generalize the principle of virtual work such that

$$[u^{IV} + \eta \dot{u}^{IV} + Q(1-x)u'' + \beta\dot{u} + \ddot{u}]\delta v$$

$$+ [v^{IV} - \eta \dot{v}^{IV} + \{Q(1-x)v\}'' + \ddot{v} - \beta\dot{v}]\delta u = 0. \tag{17}$$

Integrating Eq. (17) over the interval (0,1) and between the two time limits t_1 and t_2, and imposing end conditions of Eqs. (11) and (14), a generalized variational principle can be written [9] as the following

$$\delta \int_{t_1}^{t_2} L\, dt = 0, \tag{18}$$

with

$$L = T - V - V_d, \tag{19}$$

and

$$T = \frac{1}{2}\int_0^1 \dot{u}\,\dot{v}\,dx, \tag{20}$$

$$V = \frac{1}{2}\int_0^1 \{u''v'' - Q[(1-x)v]'u'\}dx, \tag{21}$$

$$V_d = \frac{1}{4} \int_0^1 \{\eta(v''\dot{u}'' - u''\dot{v}'') + \beta(v\dot{u} - u\dot{v})\}dx, \tag{22}$$

can be referred to as the generalized kinetic energy, potential energy and dissipative potential resepctively.

Similar results can be obtained for Beck's problem with the expression of V in Eq. (21) replaced by that of Eq. (23). Thus

$$V = \frac{1}{2} \int_0^1 [u''v'' - Q\, u'v']dx + \frac{1}{2} Q\, u'(1,t)v(1,t) \tag{23}$$

for Beck's problem and its adjoint.

The finite element formulations are based on the variational equation of (18), where L is clearly a complete functional. In comparison, the work by Barsoum [11] and that by Mote [12,13] are based on an extended Hamilton's principle of the form

$$\delta \int_{t_1}^{t_2} L_1\, dt + \int_{t_1}^{t_2} \delta W\, dt = 0 \tag{24}$$

where L_1 is a complete functional but δW_1, which is the virtual work done by nonconservative forces, cannot be expressed as the variation of a path-independent functional. Using a different approach, Anderson [36] has shown that the adjoint variational principle (18) can lead to more accurate numerical solutions than does the variational expression of (24).

3. FINITE ELEMENT FORMULATIONS

In the application of the finite element method, the column being analyzed is divided into L segments (elements) as shown in Figure 1(c). In conjunction with the variational principle of Eq. (18), the integrations in Eqs. (20), (21) and (22) are replaced by summations of the quantities in all the elements. (The following derivation is carried out for Leipholz' problem. Similar outline of

the procedure for Beck's problem will not be repeated here [37].) Thus,

$$T = \sum_{i=1}^{L} \frac{1}{2} \int_{\ell_{i-1}}^{\ell_i} \dot{u}^{(i)}\dot{v}^{(i)}dx \tag{25}$$

$$V = \sum_{i=1}^{L} \frac{1}{2} \int_{\ell_{i-1}}^{\ell_i} \{u''^{(i)}v''^{(i)} - Q[(1-x)v^{(i)}]' u'^{(i)}\}dx \tag{26}$$

and

$$V_d = \sum_{i=1}^{L} \frac{1}{2} \int_{\ell_{i-1}}^{\ell_i} [\eta(v''^{(i)}\dot{u}''^{(i)} - u''^{(i)}\dot{v}''^{(i)}) + \beta(v\dot{u} - u\dot{v})]dx \tag{27}$$

where $\ell_o = 0$, $\ell_L = 1$; and the superscript (i) denotes the quantities in the i-th element.

We effect the following change of independent variable. Let

$$\zeta = \frac{1}{\ell_i-\ell_{i-1}} (x - \ell_{i-1}) = L(x - \frac{i-1}{L}) \tag{28}$$

In Eq. (28) and in the sequel, we have assumed that all the elements are of the same length. Consequently,

$$\frac{d\zeta}{dx} = L, \qquad \frac{d^2\zeta}{dx^2} = 0 \tag{29}$$

For some function f(x),

$$\frac{df}{dx} = L\frac{df}{d\zeta}, \quad \frac{d^2f}{dx} = L^2 \frac{d^2f}{d\zeta^2} \tag{30}$$

where the notation

$$f(x) = f[x(\zeta)] = f(\zeta)$$

has been adapted for simplicity.

Using the new independent variable ζ, Eqs. (25), (26) and (27) become the following:

$$T = \frac{1}{2L} \sum_{i=1}^{L} \int_{o}^{1} \dot{u}^{(i)} \dot{v}^{(i)} d\zeta \tag{25a}$$

$$V = \frac{1}{2} \sum_{i=1}^{L} \int_{o}^{1} \{L^3 u''^{(i)} v''^{(i)}$$

$$+ Q[u'^{(i)} v^{(i)} - (L-i+1-\zeta) u'^{(i)} v'^{(i)}]\} d\zeta \tag{26a}$$

and

$$V_d = \frac{1}{4} \sum_{i=1}^{L} \int_{o}^{1} [\eta L^3 (v''^{(i)} \dot{u}''^{(i)} - u''^{(i)} \dot{v}''^{(i)})$$

$$+ \frac{\beta}{L} (v^{(i)} \dot{u}^{(i)} - u^{(i)} \dot{v}^{(i)})] d\zeta. \tag{27a}$$

We shall use a polynomial to approximate the displacement field in an element. The displacement and its first derivative (slope) will be required to be continuous between the two adjacent elements. Hence there are four degrees of freedom for each element and the polynomial shall be cubic. In terms of a Rayleigh-Ritz approximation, the coordinate function is chosen to be a piecewise analytic function.

It is convenient to let

$$\left.\begin{aligned} u^{(i)}(\zeta,t) &= a^T(\zeta)\, X(t) = a^T(\zeta)\, U^{(i)} e^{\lambda t} \\ &\text{and} \\ v^{(i)}(\zeta,t) &= a^T(\zeta)\, Y(t) = a^T(\zeta)\, V^{(i)} e^{\lambda t} \end{aligned}\right\} \tag{31}$$

where λ is the usual eigenvalue parameter,

$$a^T(\zeta) = \{a_1(\zeta) \quad a_2(\zeta) \quad a_3(\zeta) \quad a_4(\zeta)\} \tag{32}$$

is the displacement function vector and

$$\left.\begin{aligned} U^{T(i)} &= \{U_1^{(i)} \quad U_2^{(i)} \quad U_3^{(i)} \quad U_4^{(i)}\} \\ V^{T(i)} &= \{V_1^{(i)} \quad V_2^{(i)} \quad V_3^{(i)} \quad V_4^{(i)}\} \end{aligned}\right] \tag{33}$$

are the generalized displacement vectors in the i-th element, and the superscript T denotes the transpose of a matrix (or a vector). When we take $U_1^{(i)}$, $U_2^{(i)}$ to be the displacement and the slope respectively at the left end (see Figure 1(c)) of i-th element and take $U_3^{(i)}$, $U_4^{(i)}$ to be the same at the right end (same for the vector $V^{(i)}$), it is easy to see that $a(\zeta)$ must take the following form:

$$\begin{aligned} a^T(\zeta) &= \{a_1(\zeta) \quad a_2(\zeta) \quad a_3(\zeta) \quad a_4(\zeta)\} \\ &= \{1-3\zeta^2 + 2\zeta^3 \quad \zeta-2\zeta^2+\zeta^3 \quad 3\zeta^2-2\zeta^3 \quad -\zeta^2+\zeta^3\}. \end{aligned} \tag{34}$$

Using Eqs. (28)-(34) in the generalized energy expressions Eqs. (25a), (26a) and (27a), and carrying out the variation of Eq. (18), we obtain the following variational equation for the descrete system:

$$\begin{aligned} &\delta \int_{t_1}^{t_2} L\, dt \\ &= \frac{1}{2} \int_{t_1}^{t_2} \sum_{i=1}^{L} e^{2\lambda t}\{\delta U^{T(i)}\{\lambda^2 \frac{\underline{A}}{L} - \lambda[\eta L^3 \underline{C} + \beta \frac{\underline{A}}{L} \\ &+ L^3\underline{C} + Q[\underline{E} + \underline{F} - (L - i+1)\underline{B}]\}V^{(i)} \\ &+ \delta V^{T(i)}\{\lambda^2 \frac{\underline{A}}{L} + \lambda[\eta L^3 \underline{C} + \beta \frac{\underline{A}}{L}] \\ &+ L^3\underline{C} + Q[\underline{E} + \underline{F}^T - (L - i+1)\underline{B}]U^{(i)}\}\}dt = 0 \end{aligned} \tag{35}$$

where $\underline{A}$, $\underline{B}$, $\underline{C}$ and $\underline{F}$ are constant matrices whose definition and numerical values are given in the Appendix. It is noted in the above mentioned matrices only $\underline{F}$

is not symmetric. In Eq. (35), $\delta U^{(i)}$ and $\delta V^{(i)}$ are independent of each other. Since we are interested only in the solution of $U^{(i)}$, we can set

$$\delta U^{(i)} = 0, \qquad i = 1,\ldots.L. \tag{36}$$

Eq. (35) now becomes

$$\int_{t_1}^{t_2} \sum_{i=1}^{L} e^{2\lambda t}\, \delta V^{T(i)}\{\lambda^2 \frac{A}{L} + \lambda[\eta L^3 \underline{C} + \beta \frac{A}{L}] + L^3 \underline{C} + Q[\underline{E} + \underline{F}^T - (L-i+1)\underline{B}]\} U^{(i)} dt = 0 \tag{37}$$

Up to this point, the matrices involved are referred to the individual elements with a size of 4x4. Next we must form the overall matrix equation and eliminate the redundant unknowns. To accomplish this, the boundary conditions and the continuity conditions must be applied. In Leipholz' problem, the boundary conditions for the original and the adjoint problem has the same form as the following (see Reference [37] for a discussion on the adjoint boundary conditions in Beck's problem.)

$$\left.\begin{aligned} \begin{Bmatrix} U_1^{(1)} \\ U_2^{(1)} \end{Bmatrix} &= \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \\ \begin{Bmatrix} U_3^{(L)} \\ U_4^{(L)} \end{Bmatrix} &= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} U_1^{(L)} \\ U_2^{(L)} \end{Bmatrix}. \end{aligned}\right\} \tag{38}$$

The continuity conditions are also the same for the original and the adjoint problems:

$$\left.\begin{aligned} U_3^{(i-1)} &= U_1^{(i)} \\ U_4^{(i-1)} &= U_2^{(i)} \end{aligned}\right\} \quad i = 2,3,\ldots.,L. \tag{39}$$

Introducing new vectors with independent elements

$$U^T = \{U_3^{(1)} \quad U_4^{(1)} \quad U_3^{(2)} \quad U_4^{(2)} \ldots\ldots U_3^{(L-1)} \quad U_4^{(L-1)}\}$$
$$V^T = \{V_3^{(1)} \quad V_4^{(1)} \quad V_3^{(2)} \quad V_4^{(2)} \ldots\ldots V_3^{(L-1)} \quad V_4^{(L-1)}\} \tag{40}$$

and using the conditions of Eqs. (38) and (39), Eq. (37) can be rewritten as

$$\int_{t_1}^{t_2} e^{2\lambda t}\, \delta V^T K\, U\, dt = 0 \tag{40}$$

where

$$K = \lambda^2 R + \lambda S + T \tag{41}$$

is a constant matrix of 2(L-1) x 2(L-1) and R, S and T are also given in the Appendix.

Since now the vector δV in Eq. (40) is arbitrary and all its elements are independent, we obtain the final matrix eigen-equation

$$K \; U$$
$$= (\lambda^2 R + \lambda S + T)U = 0 \tag{42}$$

which is to be solved next.

4. METHODS OF COMPUTATION

Because dissipative systems possess complex eigenvalues in general, let

$$\lambda = \lambda_1 + i\,\lambda_2, \qquad i = \sqrt{-1} \tag{43}$$

denote the eigenvalue of the system, where λ_1 and λ_2 are real numbers. When λ_1 is positive and $\lambda_2 \neq 0$, the system loses stability by flutter, i.e. oscillations with increasing amplitude. For each set of damping parameters β and η, the critical load Q_{cr} is obtained when λ_1 has just changed sign from negative to positive. In this study, two methods were used in extracting the eigenvalues.

They are described briefly here.

A. Standard QR Algorithm [38,39] -

Three steps are required in using this approach:

1. To transform Eq. (55) into the standard eigenvalue equation

$$(M - \lambda I)U = 0 \tag{44}$$

where

$$M = \begin{bmatrix} \emptyset & I \\ -P^{-1}R & -P^{-1}Q \end{bmatrix} \tag{45}$$

$$U = \begin{Bmatrix} W \\ \lambda W \end{Bmatrix} \tag{46}$$

and $\emptyset$, I are zero and unit matrices, respectively.

2. To transform the matrix M into Hessenberg's form.

3. To extract all the eigenvalues from a Hessenberg's matrix using the QR iteration. Steps (2) and (3) are accomplished by two Standard Subroutines provided in the IBM System/360 Scientific Subroutine Package, i.e. HSBG and ATEIG, respectively.

B. Rosenbrock's Method of Iteration [40] -

This is a trial and error method using the route along which the function decreases most rapidly.

In terms of computer time, Method B is less efficient than Method A. In applying Method B, there is a great tendency to converge to the wrong eigenvalue. However, once located correctly, the eigenvalue can be determined with more accuracy than Method A.

5. RESULTS AND DISCUSSION

Calculation for critical loads Q_{cr} are performed for the range of damping parameters which is both of practical interest and sensitive in stabilizing or destabilizing effects. Results are shown in Figures 2 through 10.

In Figure 2, the two lowest branches of the frequency curves are shown for both the Leipholz' and the Beck's problem without either internal or external damping. For small load Q, λ is a purely imaginary number ($\lambda_1 = 0$, $\lambda_2 = \Omega$ in the notation used in the previous section.) As the value of Q increases, eventually the two lowest branches coincide and Q_{cr} is reached, beyond which λ_1 becomes positive and the column will fail by flutter. In Figure 3 through Figure 10 are presented Q/π^2 vs. the damping parameter β or η. These curves have shown that for the range $0<\beta<2.0$ and $0<\eta<0.2$, the damping effect is extremely sensitive. In this range, the internal damping has a destabilizing effect as can be seen from Figures 3, 4, 7 and 8. The internal damping can also have stabilizing effect in certain ranges, e.g. the range of $0<\beta<5.0$ and $0.1<\eta<0.5$ for Leipholz' problem as shown in Figure 4. On the other hand, the external damping always has a stabilizing effect which is very sensitive when both parameters are small. This is seen in Figures 5, 6, 9 and 10.

It may be of some interest to note in Figure 4 for Leipholz problem that when $\beta = 0$ and η approaches zero, the limit of Q_{cr} is $2.32\pi^2$ while in Figure 5, when η has been set to zero to begin with, Q_{cr} has a value of $3.93\pi^2$. For Beck's problem similar situation is observed in Figures 8 and 10 with $Q_{cr}\Big|_{\eta\to 0}^{\beta=0} = 1.10\pi^2$ and $Q_{cr}\Big|_{\eta=0}^{\beta=0} = 2.03\pi^2$. These are the cases when the solution of the limit of the problem is not the same as the limit of the solution of the original problem — a phenomena known as the destabilizing effect of damping first noted by Ziegler [41].

A few words about the convergence of the data are in order here. Both Barsoum [11] and Mote [12] have shown that for similar problems and using the same displacement functions, the finite element solutions using eight segements converge to within 1% of the known exact solutions. Eight is the number used in this analysis. The solutions of Beck's problem presented here for external damping alone show excellent agreement with the data obtained by Plaut and Infante [27]. In comparing with the solutions for Leipholz' problem without damping obtained by McGill [42] using a two-term Galerkin's procedure and those obtained by Anderson and Walter for both Leipholz' and Beck's problem with both using a three-term Ritz approximation, the data here show general agreement. Good agreement has also been established between the data obtained here and those by Dzydlo and Solarz [28] from the numerical calculations based on the exact frequency equation.

APPENDIX

Definitions and values of some matrices appeared in Section 3:

$$\underline{A} = \int_0^1 a(\zeta)\, a^T(\zeta) d\zeta$$

$$= \begin{bmatrix} -\frac{1}{5}+\frac{4}{7} & & & \text{(SYMMETRIC)} \\ \frac{3}{5}-\frac{5}{6}+\frac{2}{7} & -\frac{1}{3}+\frac{1}{5}+\frac{1}{7} & & \\ \frac{1}{2}+\frac{1}{5}-\frac{4}{7} & \frac{3}{4}-\frac{3}{5}+\frac{1}{6}-\frac{2}{7} & -\frac{1}{5}+\frac{4}{7} & \\ -\frac{3}{4}+\frac{3}{5}-\frac{1}{6}+\frac{2}{7} & -\frac{3}{4}+\frac{3}{5}+\frac{1}{7} & -\frac{3}{5}+\frac{5}{6}-\frac{2}{7} & -\frac{1}{3}+\frac{1}{5}+\frac{1}{7} \end{bmatrix} \tag{A-1}$$

$$\underline{B} = \int_0^1 a'(\zeta)\, a^T(\zeta) d\zeta$$

$$= \begin{bmatrix} 36\left(\frac{1}{5}-\frac{1}{6}\right) & & & \text{(SYMMETRIC)} \\ -6\left(\frac{1}{3}+\frac{1}{4}-\frac{3}{5}\right) & \frac{1}{3}-\frac{1}{5} & & \\ -36\left(\frac{1}{5}-\frac{1}{6}\right) & 6\left(\frac{1}{3}+\frac{1}{4}-\frac{3}{5}\right) & 36\left(\frac{1}{5}-\frac{1}{6}\right) & \\ -6\left(\frac{1}{3}+\frac{1}{4}-\frac{3}{5}\right) & -\left(\frac{1}{5}-\frac{1}{6}\right) & 6\left(\frac{1}{3}+\frac{1}{4}-\frac{3}{5}\right) & \frac{1}{3}-\frac{1}{5} \end{bmatrix} \tag{A-2}$$

$$\underline{C} = \int_0^1 a''(\zeta)\, a^{T''}(\zeta) d\zeta = \begin{bmatrix} 12 & & & \text{(SYMMETRIC)} \\ 6 & 4 & & \\ -12 & -6 & 12 & \\ 6 & 2 & -6 & 4 \end{bmatrix} \tag{A-3}$$

$$\underline{E} = \int_0^1 \zeta\, a'(\zeta)\, a^{T'}(\zeta) d\zeta$$

$$= \begin{bmatrix} 36\left(\frac{1}{4}-\frac{2}{5}+\frac{1}{6}\right) & & & \text{(SYMMETRIC)} \\ \frac{1}{2}-\frac{2}{5} & \frac{1}{5}-\frac{1}{6} & & \\ -\frac{3}{5} & -\frac{1}{10} & \frac{3}{5} & \\ 0 & -\frac{1}{60} & 0 & \frac{2}{15} \end{bmatrix} \tag{A-4}$$

$$\underline{F} = \int_0^1 a'(\zeta)\, a^T(\zeta)\, d\zeta$$

$$= \begin{bmatrix} -\frac{1}{2} & -\frac{1}{10} & -\frac{1}{2} & \frac{1}{10} \\ \frac{1}{10} & -\frac{1}{2}+\frac{1}{6} & -\frac{1}{10} & \frac{1}{60} \\ \frac{1}{2} & \frac{1}{10} & \frac{1}{2} & -\frac{1}{10} \\ -\frac{1}{10} & -\frac{1}{60} & \frac{1}{10} & 0 \end{bmatrix} \tag{A-5}$$

$$R =$$

$$\begin{bmatrix}
R_{33}^{(1)}+R_{11}^{(2)} & R_{34}^{(1)}+R_{12}^{(2)} & R_{13}^{(2)} & R_{14}^{(2)} & 0 & \cdots & \cdots & \cdots \\
R_{43}^{(1)}+R_{21}^{(2)} & R_{44}^{(1)}+R_{22}^{(2)} & R_{23}^{(2)} & R_{24}^{(2)} & 0 & \cdots & \cdots & \cdots \\
R_{31}^{(2)} & R_{32}^{(2)} & R_{33}^{(2)}+R_{11}^{(3)} & R_{34}^{(2)}+R_{12}^{(3)} & & & & \\
R_{41}^{(2)} & R_{42}^{(2)} & R_{43}^{(2)}+R_{21}^{(3)} & R_{44}^{(2)}+R_{22}^{(3)} & & & & \\
 & & & \ddots & & & & \\
0 & 0 & & & R_{33}^{(L-2)}+R_{11}^{(L-1)} & R_{34}^{(L-2)}+R_{12}^{(L-1)} & R_{13}^{(L-1)} & R_{14}^{(L-1)} \\
\vdots & \vdots & & & R_{43}^{(L-2)}+R_{21}^{(L-1)} & R_{44}^{(L-2)}+R_{22}^{(L-1)} & R_{23}^{(L-1)} & R_{24}^{(L-1)} \\
\vdots & \vdots & & & R_{31}^{(L-1)} & R_{32}^{(L-1)} & R_{33}^{(L-1)}+\bar{R}_{11} & R_{34}^{(L-1)}+\bar{R}_{12} \\
\vdots & \vdots & & & R_{41}^{(L-1)} & R_{42}^{(L-1)} & R_{43}^{(L-1)}+\bar{R}_{21} & R_{44}^{(L-1)}+\bar{R}_{22}
\end{bmatrix} \tag{A-6}$$

with

$$\left.\begin{aligned}
\bar{R}_{11} &= R_{11}^{(L)} + R_{13}^{(L)} + R_{31}^{(L)} + R_{33}^{(L)} \\
\bar{R}_{12} &= R_{12}^{(L)} + R_{13}^{(L)} + R_{14}^{(L)} + R_{32}^{(L)} + R_{33}^{(L)} + R_{34}^{(L)} \\
\bar{R}_{21} &= R_{21}^{(L)} + R_{31}^{(L)} + R_{41}^{(L)} + R_{23}^{(L)} \div R_{33}^{(L)} + R_{43}^{(L)} \\
\bar{R}_{22} &= R_{22}^{(L)} + R_{32}^{(L)} + R_{42}^{(L)} + R_{23}^{(L)} + R_{33}^{(L)} + R_{43}^{(L)} + R_{24}^{(L)} + R_{34}^{(L)} + R_{44}^{(L)}
\end{aligned}\right\} \tag{A-7}$$

The expressions for S and T are obtained by replacing the letter R with the letter S and T respectively in Eqs. (A-6) and (A-7) wherever R appears. Furthermore, $R^{(i)}$, $S^{(i)}$ and $T^{(i)}$ are defined below.

$$R^{(i)} = \frac{\underset{\sim}{A}}{L} \tag{A-8}$$

$$S^{(i)} = \eta\, L^3\, \underset{\sim}{C} + \beta \frac{\underset{\sim}{A}}{L} \tag{A-9}$$

$$T^{(i)} = L^3\, \underset{\sim}{C} + Q[\underset{\sim}{E} + \underset{\sim}{F}^T - (L - i + 1)\underset{\sim}{B}] \tag{A-10}$$

where i = 1,2,....,L.

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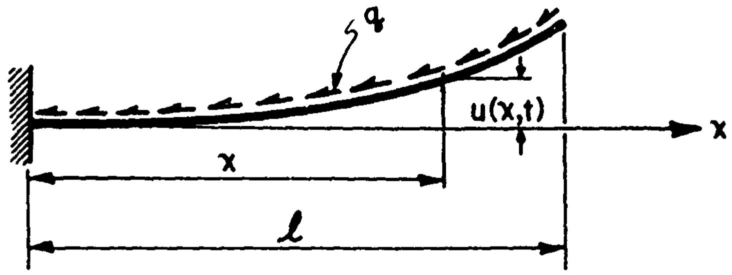
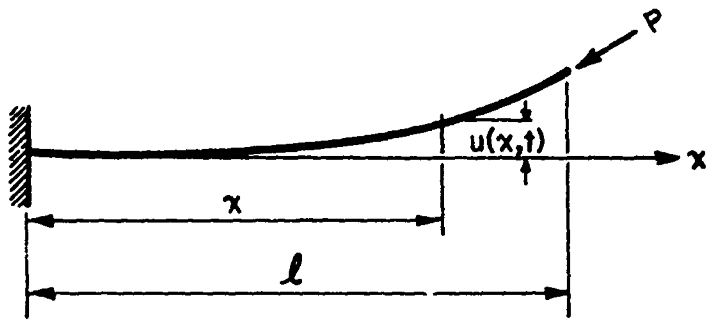
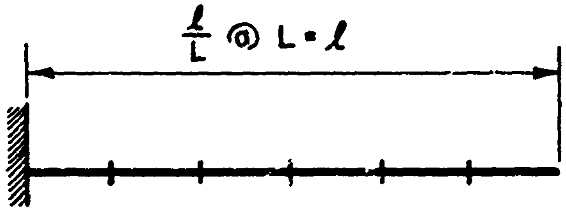


(a) LEIPHOLZ' PROBLEM



(b) BECK'S PROBLEM



(c) FINITE ELEMENT MODEL

Figure 1.

PROBLEM CONFIGURATIONS AND THE FINITE ELEMENT IDEALIZATION

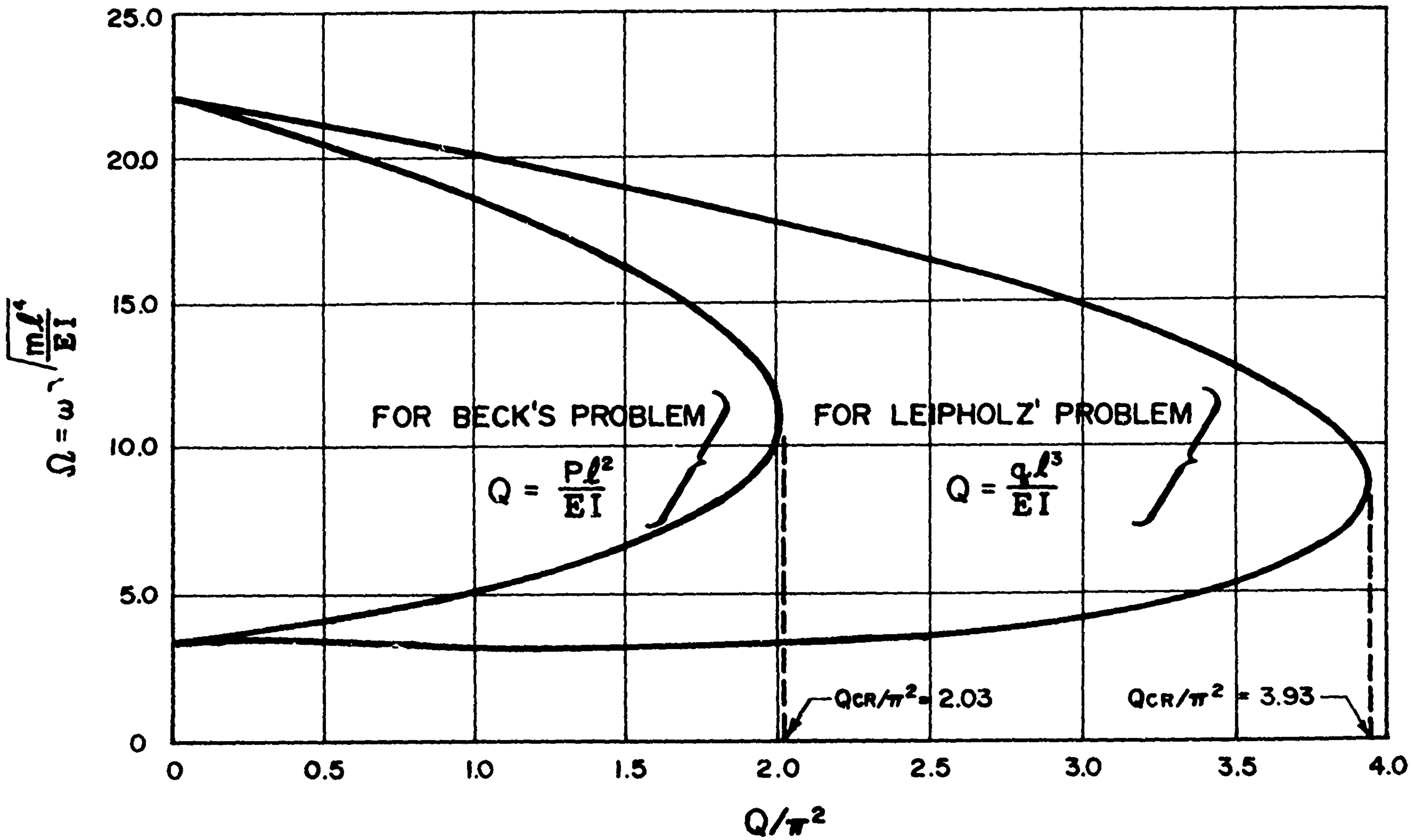


Figure 2.

FREQUENCY VS. LOAD FOR THE TWO LOWEST MODES OF LEIPHOLZ' AND BECK'S PROBLEM WITHOUT DAMPING

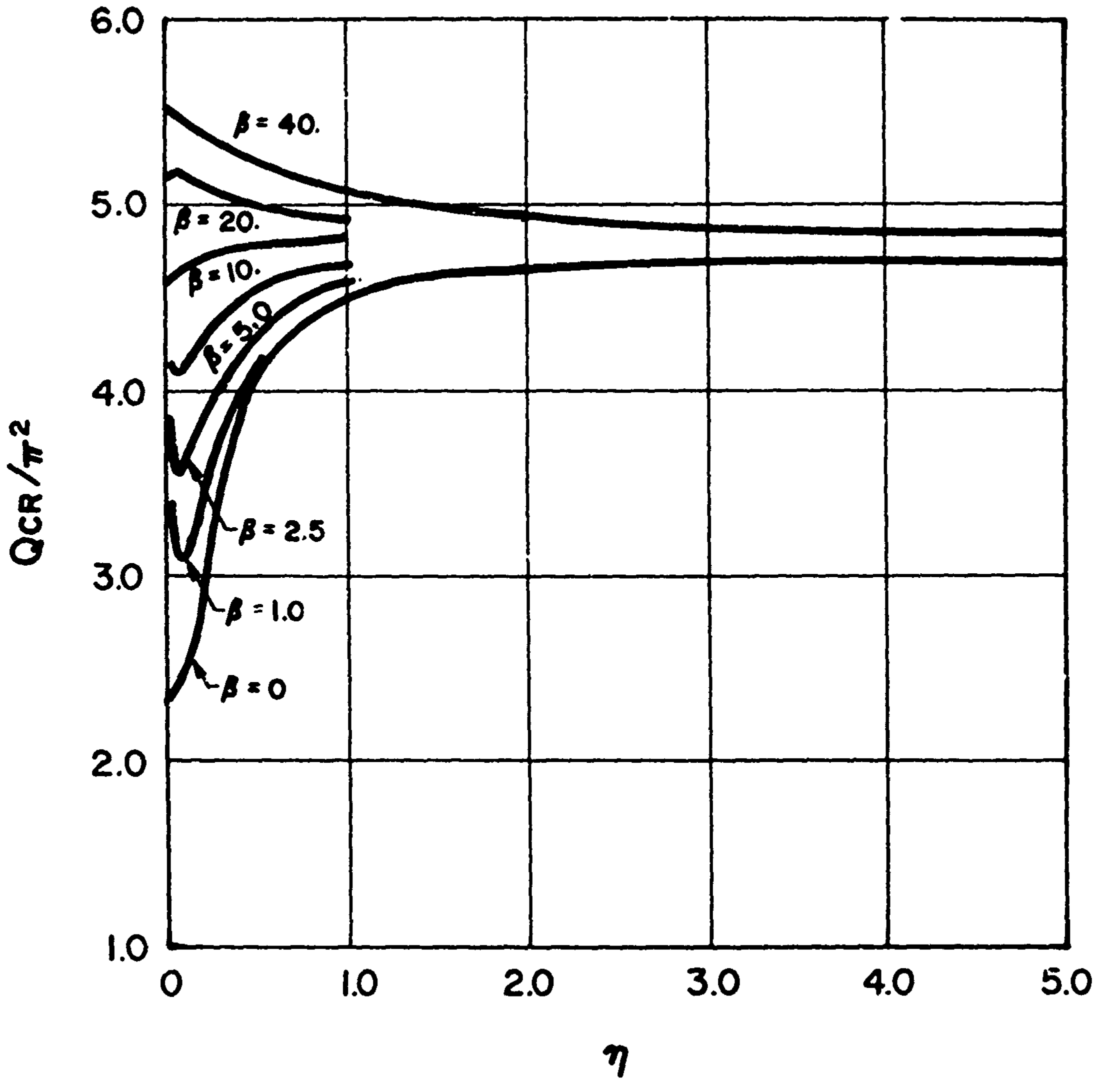


Figure 3.

LEIPHOLZ' PROBLEM: Q_{CR}/π^2 VS. η

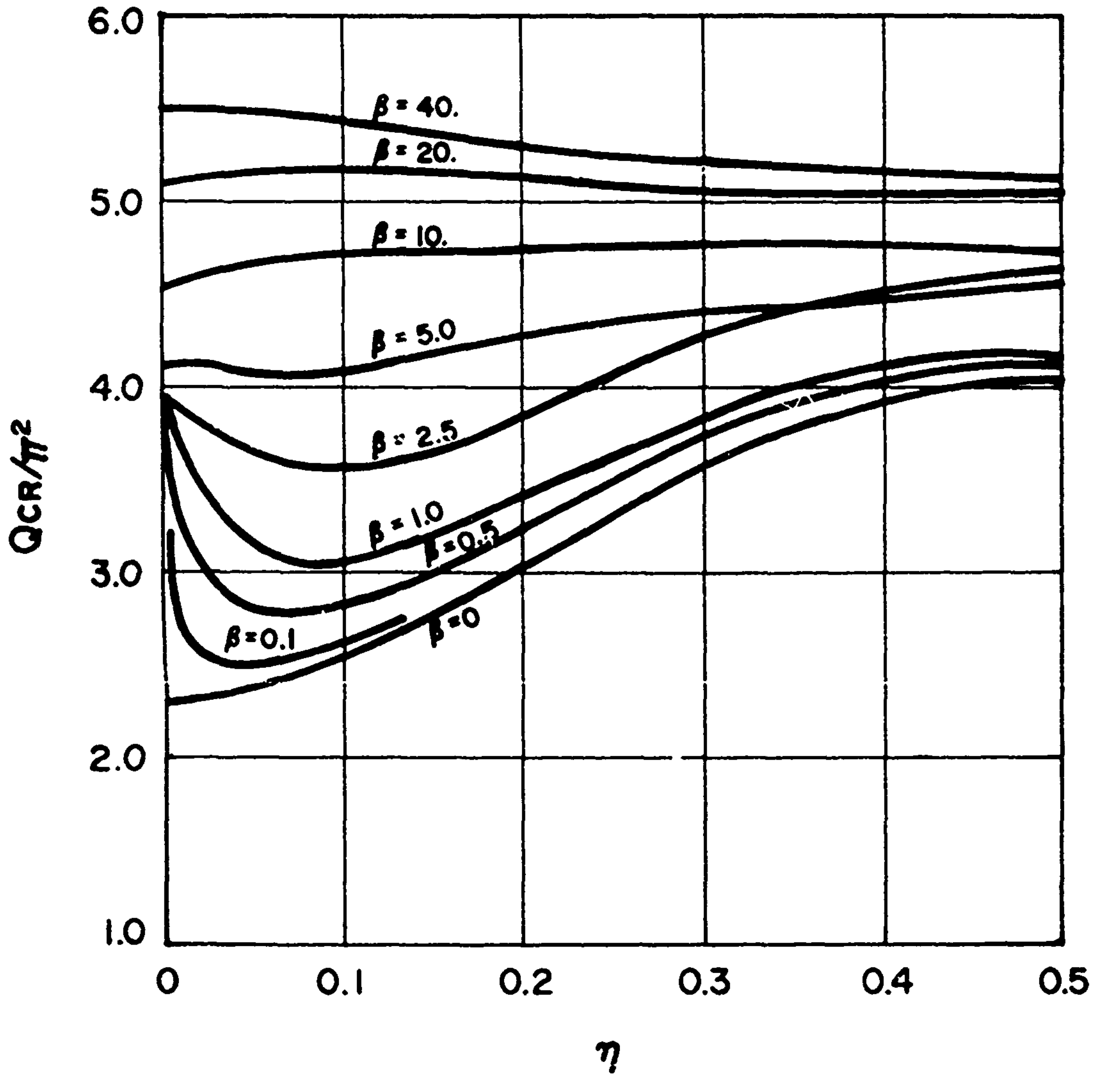


Figure 4.

LEIPHOLZ' PROBLEM: Q_{CR}/π^2 VS. SMALL η ($\eta \leq 0.5$)

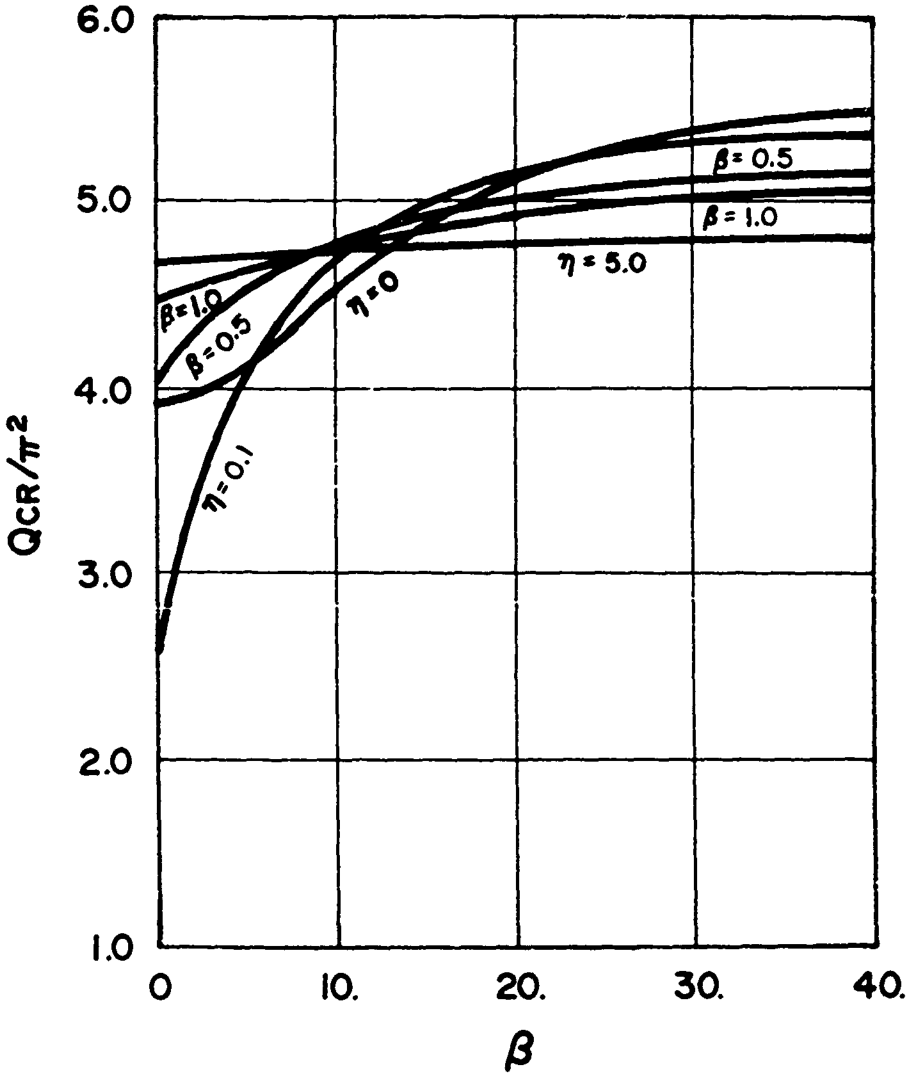


Figure 5.

LEIPHOLZ' PROBLEM: Q_{CR}/π^2 VS. β

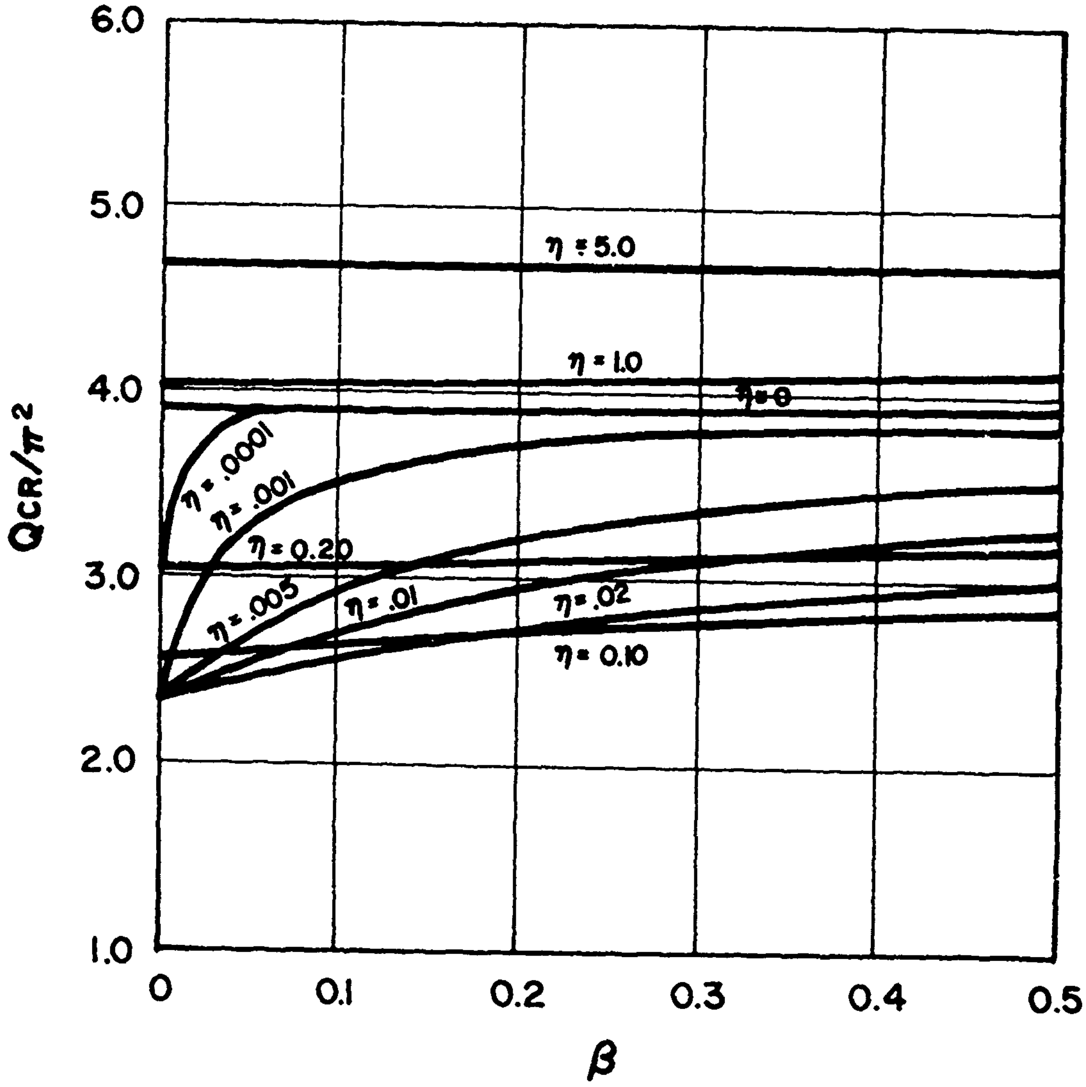


Figure 6.

LEIPHOLZ' PROBLEM: Q_{CR}/π^2 VS SMALL β
($\beta \leq 0.5$)

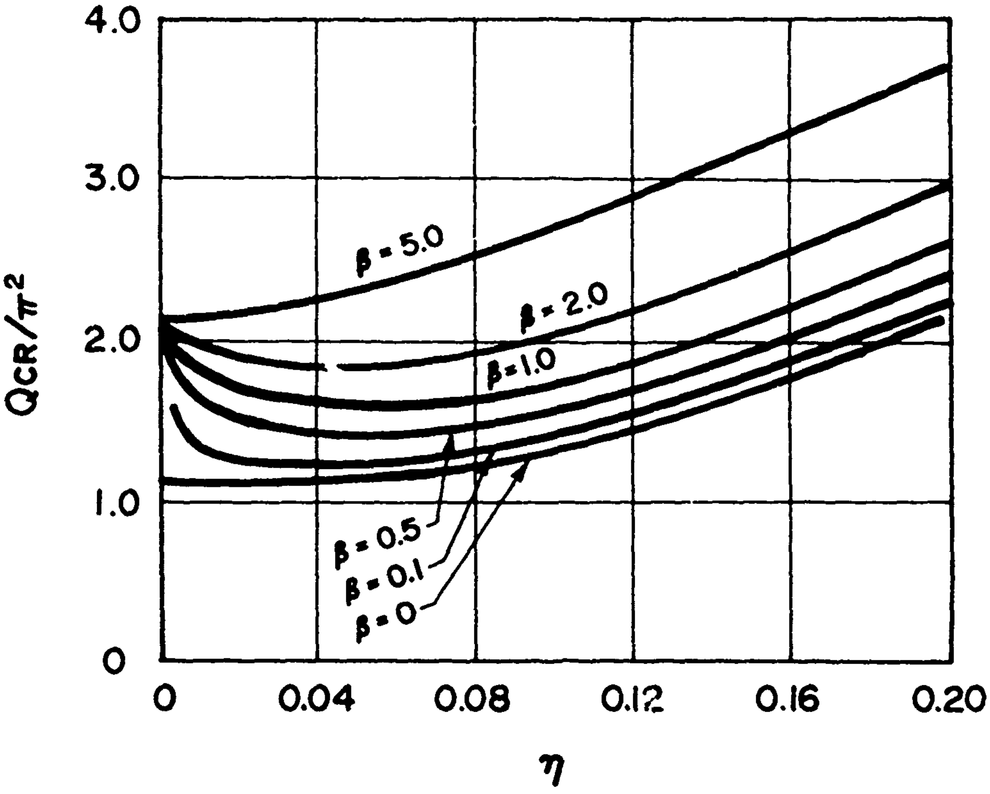


Figure 7.

BECK'S PROBLEM: Q_{CR}/π^2 VS. η

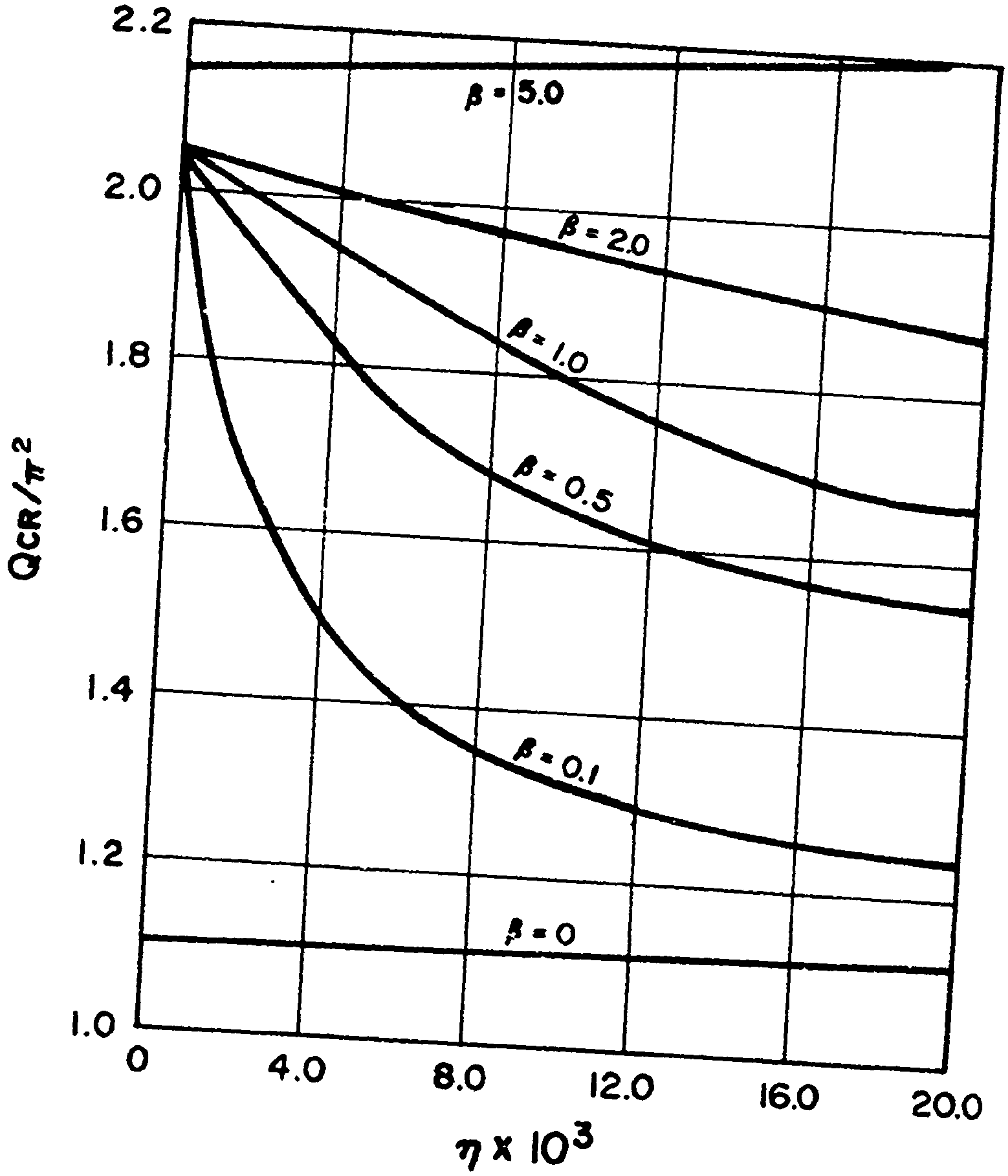


Figure 8.

BECK'S PROBLEM: Q_{CR}/π^2 VS. SMALL η
($\eta \leq 0.02$)

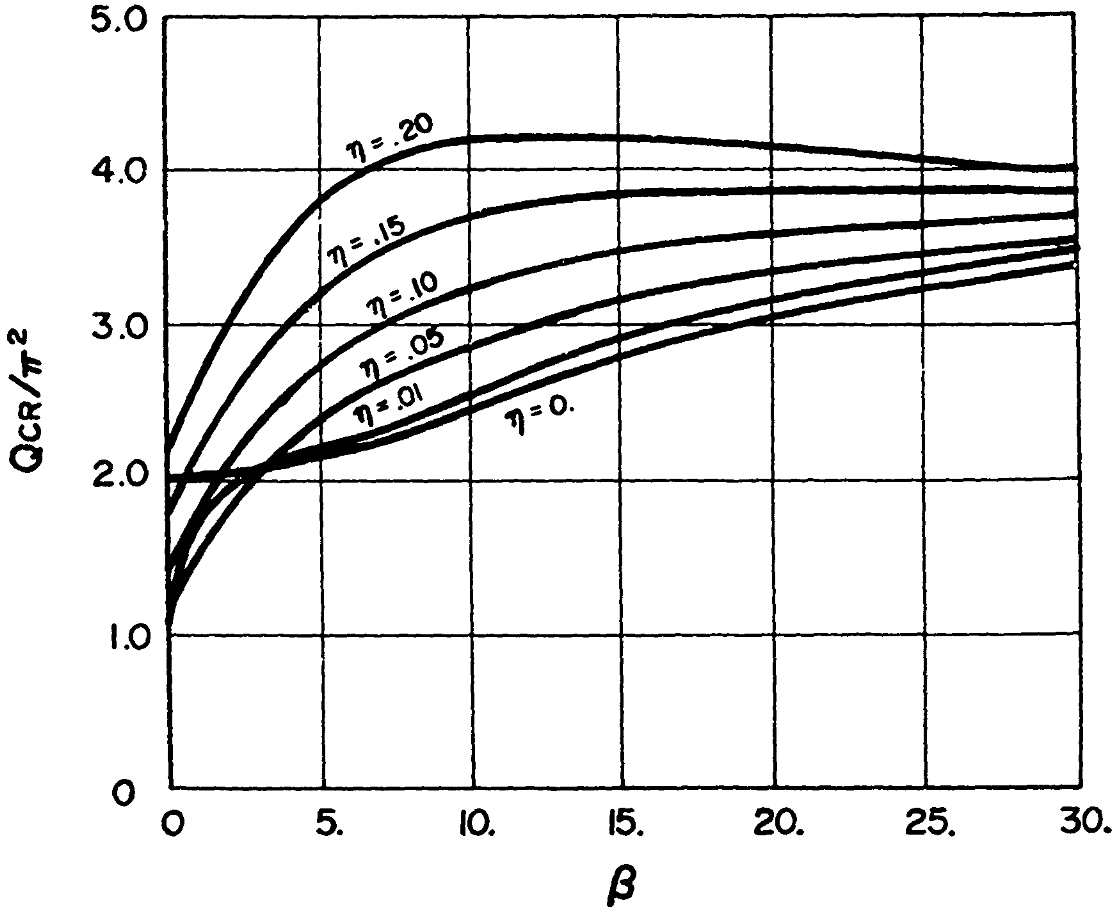


Figure 9.

BECK'S PROBLEM: Q_{CR}/π^2 VS. β

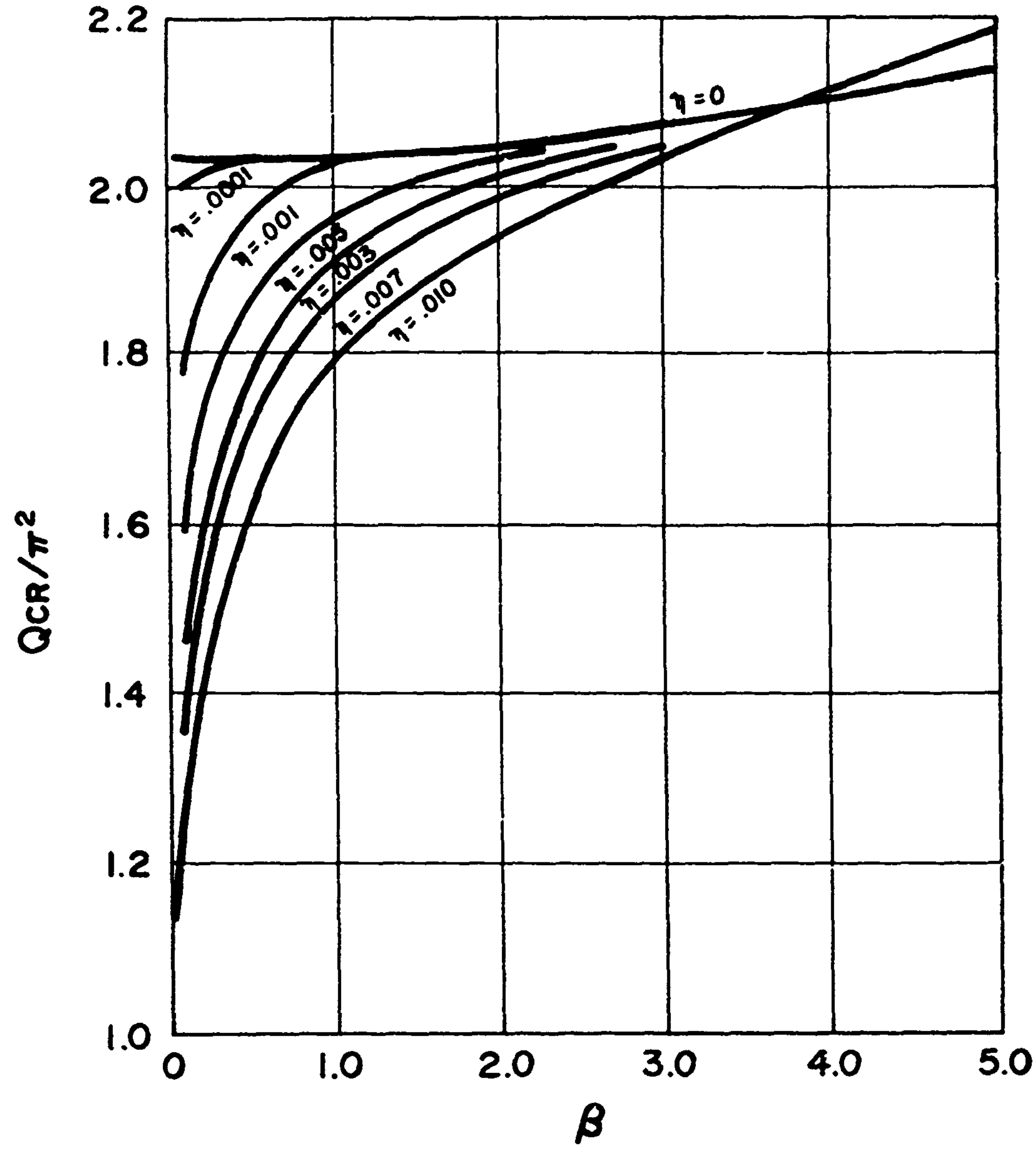


Figure 10.

BECK'S PROBLEM: Q_{CR}/π^2 VS. SMALL β
($\beta \leq 5.0$)